READING GROUP: ALGEBRAIC K-THEORY
SUMMER TERM 2026

NIKLAS ARPPE

The plan for this reading group is to provide a modern introduction to algebraic K-theory using the language of ∞-categories and higher algebra. Our main source will be the lecture notes by Hilman-McCandless [HMC24] from the lecture course taught in the summer term 2024. You can find these lecture notes here, or alternatively here (the latter ones contain solutions to several of the exercises). Another good source is the lecture notes by Winges, which you can find here.

We will begin by recalling basic concepts from higher category theory, such as localizations, presentability, stability, $\mathbb{E}_\infty$-groups and -monoids, and symmetric monoidal ∞-categories. After that we can get started with the actual contents; that is, different forms of (algebraic) K-theory. In particular, we will focus on the K-theory of stable ∞-categories through the group completion construction, the plus construction, the Q-construction, and the S-construction. This can, of course, be applied to $\mathrm{Perf}(X)$ of a scheme X or $\mathrm{Perf}(R)$ of a ring R to recover the classical objects of interest.

Prerequisites

Although we will recall some higher category theory, it is strongly recommended that you have been exposed to this before, e.g. through the higher category theory reading group from the winter term 2025/2026. Other than this, you should be able to get by with just some basic knowledge on algebra, homotopy theory, and ordinary category theory.

Schedule

The reading group will meet on Fridays 12:15–14:00 in SemR N0.003.

No.	**Date**	**Topic**	**Speaker**
0.	Apr 14th	Preliminary meeting	Niklas
1.	Apr 24th	Crash course on higher algebra	Niklas
2.	May 8th	Group-completion K-theory	Vishal
3.	May 15th	Cyclic invariance and the plus construction	Pedro
4.	May 22nd	The Q-construction and Verdier sequences	Sam
5.	Jun 5th	Waldhausen's additivity theorem and the universal property of $\mathcal{K}$	Ostap
6.	Jun 12th	Verdier localization	Maria
7.	Jun 19th	Karoubi localization	Jordi
8.	Jun 26th	Waldhausen categories and the S-construction	Gabriel
9.	Jul 3rd	The +=S theorem and dévissage arguments	Niklas
10.	Jul 10th	The theorem of the t-heart	Philipp
11.	Jul 17th	The Blumberg–Mandell regularity localization sequence	Lewis
12.	Jul 24th	Burklund–Levy unipotent dévissage	x

Note: Subsections and page numbering always refer to [HMC24].

Talk 0: Preliminary meeting. During this meeting, we will provide an overview of the topic and assign talks. If you are interested in giving a talk, please attend the meeting. Any remaining talks will be available afterward, and you may contact me directly if you would like to present.

Talk 1: Crash course on higher algebra. We decided to combine the first two talks into one, so this will be a recap/crash course on higher algebra and localizations. It will essentially cover all of Section 2, hence the pace will be very quick.

Talk 2: Group-completion K-theory. Cover subsection 3.1 (roughly pages 33-38). Define K-theory via the group completion K-theory functor k and construct its symmetric monoidal refinement. Discuss localizations of $\mathbb{E}_\infty$-rings and prove the group completion theorem. This has relatively little content, so you can choose to flesh out the details and/or add examples from external sources.

Talk 3: Cyclic invariance and the plus construction. Cover subsections 3.2-3.3. (pages 39-51), but you may focus on 3.2. Discuss and prove the cyclic invariance condition. Feel free to skip the results on hypoabelian groups if you have issues with time. State Quillen's $+$-construction and compute the K-theory of finite sets. You may want to consult other more classical sources (e.g. [Wei13]) for more intuition on the plus construction.

Talk 4: The Q-construction and Verdier sequences. Cover subsections 4.1-4.2 (roughly pages 52-64). Define the K-theory of (small) stable ∞-categories via Quillen's Q-construction using the twisted arrow category. Define Verdier sequences and state the condition for a sequence to be a Verdier sequence (Proposition 4.2.9). Define additive and Verdier localizing functors, and discuss some criteria for these. This has a lot of material, so you will again need to pick and choose. You can definitely leave out some of the technical exercises. You may want to consult [Wei13] for intuition on the construction.

Talk 5: Waldhausen's additivity theorem and the universal property of $\mathcal{K}$. Cover subsection 4.3 (roughly pages 65-76). State and prove (feel free to only give a sketch) Waldhausen's additivity theorem, which in particular implies that our definition of K-theory takes values in connective spectra. Deduce consequences of the theorem. Prove the universal property of K-theory, i.e., that it is the initial grouplike additive functor under $(-)^{\simeq}$.

Talk 6: Verdier localization. Cover subsection 4.4 (roughly pages 77-82). Prove that K-theory is Verdier localizing. State Waldhausen's generic fibration theorem and discuss its corollaries. As this subsection is not very long, you can either choose to come up with external examples or delve deeper into the theory (cf. [HLS23] for background and references).

Talk 7: Karoubi localization. Cover subsection 4.5 (pages 83-98). Define Karoubi sequences, which are variants of Verdier sequences that e.g. are invariant under idempotent completion, and discuss how we can detect these (Proposition 4.5.9, Proposition 4.5.11). Prove that K-theory is Karoubi localizing. Construct a variant Quillen's localization sequence (Example 4.5.44). Sketch how the fact that K theory is Karoubi localizing gives us connective K-theory.

Talk 8: Waldhausen categories and the S-construction. Cover subsections 5.1-5.2 (pages 99-106). State the theorems that we will be aiming for in the next talks. Define Waldhausen categories and construct K-theory via the S-construction. Sketch the proof that the S- and Q-constructions agree (Theorem 5.2.16); for this you will want to consult [HW21, Proposition IV.8], which you can find here.

Talk 9: The $+$=S theorem and dévissage arguments. Cover subsections 5.3-5.4 (roughly pages 107-117). Prove the $+$=S theorem. This is very technical, so you will need to decide how much of it to present and in what amount of detail. Prove Quillen's dévissage theorem and sketch the proof of the resolution theorem.

Talk 10: The theorem of the *t*-heart. Cover subsections 5.5-5.6 (roughly pages 117-129). Recall the basics of t-structures on stable ∞-categories and discuss their basic properties. Prove the theorem of the t-heart relating the K-theory of the a stable ∞-category with a t-structure to the K-theory of the heart of the t-structure. If time permits, state the corresponding theorem for weight structures.

Talk 11: The Blumberg–Mandell regularity localization sequence. Cover subsections 5.7-5.8 (roughly pages 130-138). Discuss coherence and regularity of rings and generalize this to $\mathbb{E}_\infty$-ring spectra. State and prove the regularity localization sequence theorem. Construct the real and complex topological K-theory spectra and prove the Blumberg-Mandell localization theorem. Use this to construct Quillen's localization sequence.

Talk 12: Burklund–Levy unipotent dévissage. Cover subsection 5.9 (roughly pages 139-144). Define unipotent functors and discuss pasting of filtrations. You can simply state Lemma 5.9.11 so as to avoid recalling how filtered spectra yield spectral sequences. State and prove the Burklund-Levy unipotent dévissage theorem This subsection is very short, so you will want to consult the original source [BL21a]. For context, you may also want to look up the (incomplete) script [BL21b], which you can find here.

References

[BL21a] Robert Burklund and Ishan Levy, *On the K-theory of regular coconnective rings*, 2021. arXiv: 2112.14723 [math.KT].

[BL21b] ———, *Some aspects of noncommutative geometry*, Incomplete, 2021.

[HLS23] Fabian Hebestreit, Andrea Lachmann, and Wolfgang Steimle, *The localisation theorem for the K-theory of stable ∞-categories*, 2023. arXiv: 2205.06104 [math.KT].

[HMC24] Kaif Hilman and Jonas McCandless, *Introduction to algebraic K-theory*, 2024.

[HW21] Fabian Hebestreit and Ferdinand Wagner, *Algebraic and Hermitian K-theory*, 2021.

[Wei13] C.A. Weibel, *The K-book: An Introduction to Algebraic K-theory* (Graduate Studies in Mathematics). American Mathematical Society, 2013, ISBN: 9780821891322.

Universität Bonn, Bonn, D-53123
E-mail address: niklasarppe@uni-bonn.de
URL: https://niklasarppe.com/