READING GROUP: MODULI SPACES OF SHTUKAS
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The purpose of this reading group is to provide an introduction to p-adic geometry and the theory of moduli spaces of mixed-characteristic shtukas, following [SW20]. Let us briefly sketch the plan.

Let $X/\mathbf{F}_p$ be a smooth projective curve. Roughly speaking, a *shtuka* of rank n over an $\mathbf{F}_p$-scheme S consists of a rank n vector bundle $\mathcal{E}$ on $S \times_{\mathbf{F}_p} X$, together with additional data relating $\mathcal{E}$ to its Frobenius pullback. One can then define moduli stacks of shtukas, whose cohomology plays a central role in the proof of the global Langlands correspondence for function fields. Following the ideas of [Laf18], one hopes to realize local Langlands correspondences in an analogous way using moduli spaces of mixed-characteristic shtukas.

As such, we will be concerned with the classical arithmetic Langlands program while borrowing methods from the geometric Langlands program. Pursuing this point of view naturally leads to Fargues' geometrization conjecture (cf. [FS24]), which roughly asserts that the arithmetic local Langlands correspondence for a reductive group $G/\mathbf{Q}_p$ should arise from the unramified geometric Langlands correspondence on the Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF}}$.

Ideally, one would like to have a theory of shtukas over number fields. However, the resulting moduli spaces cannot be expected to live on a naive object such as $\operatorname{Spec} \mathbf{Z} \times \operatorname{Spec} \mathbf{Z}$. To remedy this, we introduce the notion of diamonds, which can be thought of roughly as playing the role of algebraic spaces in the classical setting and provide a framework for rigorously defining the objects we seek. Pursuing this further, we define mixed-characteristic shtukas and investigate their relation to the Langlands program. We will soon see that the moduli spaces in question naturally live in a category of stacks; this leads us to the definition of the v-topology, which may be viewed as an analogue of the fpqc topology in classical algebraic geometry. After an interlude on the B_{dR}^+-affine Grassmannian and G-bundles on the Fargues–Fontaine curve, we finally define moduli spaces of mixed-characteristic local shtukas and prove that they can be realized as locally spatial diamonds. As an application, we conclude by specializing the theory to local Shimura varieties and discussing their relation with Rapoport–Zink spaces.

This seminar is essential for anyone interested in the local Langlands program, but will also be useful for anyone with a general interest in arithmetic geometry.

Prerequisites

It will be absolutely necessary to have a solid understanding of algebraic geometry. Prior exposure to representation theory (in particular, the Langlands program) will certainly be helpful, but should not be necessary for all of the talks. It goes without saying that any experience with non-archimedean analytic geometry will also be very helpful.

Schedule

The reading group will meet on TBA in SemR TBA.

No.	Date	Topic	Speaker
0.	x	Preliminary meeting	Erick and Niklas
1.	x	Adic spaces	x
2.	x	Perfectoid spaces	x
3.	x	Diamonds	x
4.	x	Diamonds II	x
5.	x	Diamonds associated with adic spaces	x
6.	x	Mixed-characteristic shtukas	x
7.	x	Shtukas with one leg	x
8.	x	The v-topology	x
9.	x	The B_{dR}^{+}-affine Grassmannian	x
10.	x	Vector bundles and G-torsors on the Fargues–Fontaine curve	x
11.	x	Moduli spaces of mixed-characteristic local shtukas	x
12.	x	Local Shimura varieties	x

Syllabus

Talk 0: Preliminary meeting. The organizers will give an overview of the seminar and distribute the talks. If you are interested in giving a talk, you should attend the preliminary meeting or contact one of the organizers. Any talks remaining after the preliminary meeting will be distributed later.

Talk 1: Adic spaces. This talk will cover [SW20, Lectures 2–5]. As such, you will only have time to give a brief overview. Define the basic notions related to adic rings and adic spaces. Time permitting, discuss pre-adic spaces and analytic adic spaces. You can gloss over the technincalities related to non-sheafy Huber pairs. Try to include many examples, such as the terminal adic space $\operatorname{Spa}\mathbf{Z}$, the adic closed unit disc $\operatorname{Spa}\mathbf{Z}[T]$, and the adic affine line $\operatorname{Spa}(\mathbf{Z}[T],\mathbf{Z})$. At the very least, you should include the open unit disc $\mathbf{D}_K$ and its punctured variant $\mathbf{D}_K^*$ — these will be very important later on.

Talk 2: Perfectoid spaces. This talk will cover [SW20, Lectures 6–7]. Define perfectoid rings, fields, tilting, and give the usual examples ([SW20, Example 6.15]). Define perfectoid spaces as a subcategory of adic spaces and explain why we want to study them ([SW20, Section 7.2]). Conclude by giving a brief introduction to almost mathematics and define the étale site of perfectoid spaces.

Talk 3: Diamonds. This talk will cover [SW20, Lecture 8, Lecture 15]. Start by motivating the construction of diamonds and defining the pro-étale topology on perfectoid spaces. Define diamonds and discuss some examples, including Banach-Colmez spaces ([SW20, Section 15.2]). You may also want to take a look at [Sch26], but focus on including many examples.

Talk 4: Diamonds II. This talk will cover [SW20, Lecture 9]. Explain how we run into issues when considering pro-étale descent for perfectoid spaces and state under which assumptions it is effective ([SW20, Theorem 9.1.3]). Explain how we can enlarge the class of pro-étale morphisms to quasi-pro-étale morphisms and obtain an equivalent characterization of diamonds. Take some time to define $\operatorname{Spd}\mathbf{Q}_p$ and discuss some basic properties. Prove that the category of perfectoid spaces over $\mathbf{Q}_p$ is equivalent to the category of perfectoid spaces X of characteristic p with a map to $\operatorname{Spd}\mathbf{Q}_p$ ([SW20, Theorem 9.4.4]).

Talk 5: Diamonds associated with adic spaces. This talk will cover [SW20, Lecture 10]. Construct the functor $X \longmapsto X^\lozenge$ sending an analytic pre-adic space X over $\mathbf{Z}_p$ to a diamond $X^\lozenge$. Use this to show that there is an isomorphism of diamonds

$$\operatorname{Spd}\mathbf{Q}_p \times \operatorname{Spd}\mathbf{Q}_p \cong (\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p})^\lozenge/\underline{\mathbf{Z}_p^\times}.$$

Conclude by discussing the underlying topological space of a diamond and defining the étale site of diamonds.

Talk 6: Mixed-characteristic shtukas. This talk will cover [SW20, Lecture 11]. Recall the definition of a shtuka over a scheme, and define the corresponding notion of a *local shtuka* over an adic space. Explain the strategy for extending this definition to mixed characteristic. Recalling [SW20, Theorem 5.2.8], define mixed characteristic shtukas and note the analogy to Breuil–Kisin–Fargues modules.

Talk 7: Shtukas with one leg. This talk will cover [SW20, Lectures 12–14]. Again, you will need to heavily pick and choose what to cover. Discuss the adic spectrum $\operatorname{Spa} A_{\mathrm{inf}}$ in detail and show that $\mathcal{Y} = \operatorname{Spa} A_{\mathrm{inf}} \smallsetminus \{x_k\}$ is an adic space. Use this to define the Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF}}$ and discuss its basic properties. Prove Fargues' theorem characterizing shtukas over $\operatorname{Spa} C^\flat$ with one leg ([SW20, Theorem 14.1.1]).

Talk 8: The v-topology. This talk will cover [SW20, Lectures 17–18]. Define the v-topology on the category of perfectoid spaces, mimicking the fpqc-topology on the category of schemes. Prove that representable presheaves are v-sheaves ([SW20, Corollary 17.1.5]) and state that, more generally, diamonds are v-sheaves. Mention several examples of v-stacks. To single out v-sheaves that are reasonable to work with, introduce small and spatial v-sheaves, making sure to mention why these definitions are useful (cf. [SW20, Theorem 17.3.9]). Quickly discuss morphisms of v-sheaves, and conclude by mentioning how some "non-analytic" objects give rise to v-sheaves [SW20, Lecture 18].

Talk 9: The B^+_{dR}-affine Grassmannian.[1] This talk will cover [SW20, Lecture 19]. For a reductive group G defined over an algebraically closed non-archimedean field $C/\mathbf{Q}_p$, define the B^+_{dR}-affine Grassmannian Gr_G, as well as the associated loop groups LG and L^+G. Define open subfunctors

$$\mathrm{Gr}_\mu \subseteq \mathrm{Gr}_{\leq\mu} \subseteq \mathrm{Gr}_G$$

for a dominant cocharacter $\mu \in X_*(T)^+$, where $T \subseteq G$ is a maximal torus, and prove that $\mathrm{Gr}_{\leq\mu}$ is a spatial diamond [SW20, Theorem 19.2.4]. Specialize to $G = \mathrm{GL}_n$ and discuss the relation to the classical theory of desingularizations of Schubert varieties [SW20, Section 19.3]. To conclude, define the flag variety $\mathscr{F}\!\ell_{G,\mu}$, and define the Bialynicki–Birula map

$$\pi_\mu \colon \mathrm{Gr}_\mu \to \mathscr{F}\!\ell^\lozenge_{G,\mu},$$

which is an isomorphism if μ is minuscule [SW20, Section 19.4].

Talk 10: Vector bundles and G-torsors on the Fargues–Fontaine curve. This talk will cover [SW20, Lecture 22]. Following the appendix to [SW20, Lecture 19], you may wish to recall different approaches to the theory of G-torsors. Relate the category of vector bundles on $\mathcal{X}_{\mathrm{FF},S}$ to the category of φ-equivariant finite projective modules over the relative Robba ring $\widetilde{\mathcal{R}}_R$. Over a geometric point $S = \operatorname{Spa}(C, C^+)$, define Kottwitz's set $B(G)$ and use this to classify G-bundles on $\mathcal{X}_{\mathrm{FF},S} = \mathcal{X}_{\mathrm{FF}}$ [SW20, Theorem 22.4.3]. Use the rest of the time to generalize this to the relative case.

[1] A significant background in representation theory and preferrably the Langlands program is necessary for this talk.

Talk 11: Moduli spaces of mixed-characteristic local shtukas. This talk will cover [SW20, Lecture 23]. For a suitable triple $(\mathcal{G}, b, \{\mu_i\}_{i=1}^m)$, define the associated moduli space of mixed-characteristic local G-shtukas

$$\mathrm{Sht}_{\mathcal{G},b,\{\mu_i\}} \to \mathrm{Spd}\,\breve{E}_1 \times_{\mathrm{Spd}\,k} \cdots \times_{\mathrm{Spd}\,k} \mathrm{Spd}\,\breve{E}_m$$

and prove that it is (representable by) a locally spatial diamond [SW20, Theorem 23.1.4]. The proof of this will occupy essentially the rest of the talk. Follow the strategy in the notes, starting with the case of no legs and then moving onto to more general cases. You may encounter notions we have not discussed (such as the Beilinson–Drinfeld Grassmannian), so you will have decide the amount of detail you go into.

Talk 12: Local Shimura varieties.[2] This talk will cover [SW20, Lecture 24]. Specialize the theory back to the case of local Shimura varieties and explain the relation with Rapoport–Zink spaces. Start by defining local Shimura data and Shimura varieties, leading with the classical definition in the global case. Recall the definition of Rapoport–Zink spaces, and that they are representable by formal schemes $\mathcal{M}_{\mathbb{X}}$ over $\mathrm{Spd}\,\breve{\mathbf{Z}}_p$ [SW20, Theorem 24.2.2]. Prove that there is a natural isomorphism

$$\mathcal{M}_{\mathbb{X},\breve{\mathbf{Q}}_p}^{\diamondsuit} \to \mathrm{Sht}_{\mathrm{GL}_n,b,\mu},$$

establishing the connection with shtukas [SW20, Theorem 24.2.5]. Time permitting, discuss EL and PEL data; however, you should rather opt to include more classical background.

References

[FS24] Laurent Fargues and Peter Scholze. *Geometrization of the local Langlands correspondence*. 2024. arXiv: 2102.13459 [math.RT]. URL: https://arxiv.org/abs/2102.13459.

[Laf18] Vincent Lafforgue. "Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale". *Journal of the American Mathematical Society* 31.3 (2018), pp. 719–891.

[Sch26] Peter Scholze. *Etale cohomology of diamonds*. 2026. arXiv: 1709.07343 [math.AG]. URL: https://arxiv.org/abs/1709.07343.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*. Vol. 207. Princeton: Princeton University Press, 2020.

[2] Prior background in the theory of classical Shimura varieties is necessary for this talk.